

# A BLOW-UP CRITERION FOR 3-D COMPRESSIBLE VISCO-ELASTICITY

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ABSTRACT. In this paper, we prove a blow-up criterion for 3D compressible visco-elasticity in terms of the upper bound of the density and the deformation tensor.

**Keywords:** 3D Visco-elasticity, compressible, blow-up criterion.

## 1. INTRODUCTION

Visco-elastic fluids exhibit the characteristics for both fluid and solid. The elastic behavior of these materials is attributed to the underlying microstructure or configurations, the deformation of the structures will cause the exchange of kinetic energy and elastic internal energy. The energy exchange is realized through a coupling of the transport of the internal elastic variables and the induced elastic stress.

To illustrate this coupled interaction, we denote  $x(t, X)$  as the Eulerian coordinate, where  $X$  is the Lagrangian coordinate of the particles. The velocity field defined on the Eulerian coordinate is

$$(1.1) \quad u(t, x) = x_t(t, X(t, x)),$$

and the deformation tensor is

$$(1.2) \quad F(t, x) = H(t, X(t, x)) = \frac{\partial x}{\partial X}(t, X),$$

which is determined by a transport equation (see [3, 8, 16]),

$$(1.3) \quad F_t + u \cdot \nabla F = \nabla u F.$$

This equation is regarded as the compatibility condition between velocity  $u$  and deformation tensor  $F$ . For homogenous, hyper-elastic and isotropic material, the action function is (see [11, 16, 17])

$$(1.4) \quad A(x) = \int_0^T \int_{\Omega_0} \frac{1}{2} \rho_0(X) |x_t(t, X)|^2 - E(H) - P(\rho(x(X, t), t)) \det H dX dt,$$

where  $\rho_0(X)$  is the density in the undeformed configuration,  $\rho$  is the density in the deformed configuration.  $\Omega_0$  is the original domain occupied by the material.  $E(H)$  is the elastic energy function.  $P(\rho)$  is the hydrostatic pressure, satisfying  $P(\rho) = C_0 \rho^\gamma$ , with  $C_0 > 0$  and the adiabatic index  $\gamma > 1$ .  $P(\rho) \det H$  represents the internal energy. Taking

a variation of  $A(x)$  with respect to the flow map  $x(t, X)$ , we get the inviscid momentum equation in Eulerian coordinates ([10])

$$(1.5) \quad \rho(u_t + u \cdot \nabla u) + \nabla P = \nabla \cdot \left( \frac{1}{\det F} \frac{\partial E(F)}{\partial F} F^T \right),$$

where  $F^T$  is the transpose of  $F$ , the term  $\frac{1}{\det F} \frac{\partial E(F)}{\partial F} F^T$  is the Cauchy-Green tensor. When we take the dissipation energy into consideration, see [1, 2], we can get the corresponding viscosity momentum equation

$$(1.6) \quad \rho(u_t + u \cdot \nabla u) + \nabla P = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u + \nabla \cdot \left( \frac{1}{\det F} \frac{\partial E(F)}{\partial F} F^T \right).$$

Combining (1.3), (1.6) and mass conservation equation, with Hookean elasticity, we get the following viscoelasticity system

$$(1.7) \quad \begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ \rho(u_t + u \cdot \nabla u) + \nabla P = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u + \nabla \cdot \left( \frac{1}{\det F} F F^T \right), \\ F_t + u \nabla F = \nabla u F. \end{cases}$$

The viscosity coefficient  $\mu, \lambda$  are constants satisfying  $\mu \geq 0, 3\lambda + 2\mu \geq 0$ .

The system (1.7), when  $F$  do not exist, it is a compressible Navier-Stokes equations. The global strong solutions existence for large initial data is still open. Therefore, there are many efforts to study the blow-up criteria. See [7, 19] and references therein.

When we take the deformation tensor  $F$  into consideration, the system (1.7) will be much more complicate, although the deformation tensor admits a transport equation which is similar to the continuity equation. The incompressible visco-elasticity has been studied by [9, 10, 11, 12, 14, 16, 17] etc. Precisely, in 2D, the local strong solutions to large data and global strong solutions to small data have been studied in [10, 11, 12, 14, 16]. The global strong solutions in 3D with small data have been presented in [12]. Subsequently, the blow-up criteria for 2D Oldroyd model have been presented in [13], a blow-up criterion in 3D with partial viscosity was proved in [15].

For the compressible case, The local strong solutions with large data and global strong solutions with small data to (1.7) were proved in [4] and [5, 18] respectively. Subsequently, Hu-wang [6], presented a blow-up criterion for local strong solutions as  $\|\nabla u\|_{L^1(0,T,L^\infty(\mathbb{R}^3))} < \infty$ . To proceed, we firstly introduce some notations as following

$$D^k(\mathbb{R}^3) = \{u \in L^1_{loc}(\mathbb{R}^3) : \|\nabla^k u\|_{L^2(\mathbb{R}^3)}\},$$

$$D_0^k(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \|\nabla^k u\|_{L^2(\mathbb{R}^3)} < \infty\}.$$

In Hu-Wang[6], they have proved the following results,

**Theorem 1.1.** (*Hu-Wang [6]*). Assume that the initial data satisfy  $0 \leq \rho_0 \in H^3(\mathbb{R}^3)$ ,  $u_0 \in D_0^1(\mathbb{R}^3) \cap D^3(\mathbb{R}^3)$ ,  $F_0 \in H^3(\mathbb{R}^3)$ ,  $\nabla \cdot (\rho_0 F_0) = 0$  and

$$-\mu \Delta u_0 - (\lambda + \mu) \nabla \operatorname{div} u_0 + A \nabla \rho_0^\gamma = \rho_0 g,$$

for some  $g \in H^1(\mathbb{R}^3)$  with  $\sqrt{\rho_0}g \in L^2(\mathbb{R}^3)$ . There exist classical solutions  $(\rho, u, F)$  to (1.7) satisfying

$$(1.8) \quad \begin{cases} (\rho, F) \in C([0, T^*], H^3(\mathbb{R}^3)), \\ u \in C([0, T^*], D_0^1(\mathbb{R}^3) \cap D^3(\mathbb{R}^3)) \cap L^2(0, T^*, D^4(\mathbb{R}^3)), \\ u_t \in L^\infty([0, T^*, D_0^1(\mathbb{R}^3)) \cap L^2(0, T^*, D^2(\mathbb{R}^3)), \\ \sqrt{\rho}u_t \in L^\infty([0, T^*, L^2(\mathbb{R}^3)), \end{cases}$$

where  $T^*$  is the maximal existence time. If  $T^* < \infty$  and  $7\mu > \lambda$ , then,

$$(1.9) \quad \lim_{T \rightarrow T^*} \int_0^T \|\nabla u\|_{L^\infty(\mathbb{R}^3)} dt = \infty.$$

Enlightened by the work of Sun-Wang-Zhang [19], in which the authors presented the blow-up criterion for a compressible Navier-Stokes in the terms of  $\|\rho\|_{L^\infty(0, T, L^\infty)}$ , we will establish a blow-up criterion in terms of the upper bounds of the density and deformation tensor for the local strong solution to the 3D compressible visco-elasticity. Our main result states as follows.

**Theorem 1.2.** Assume that  $(\rho, u, F)$  is the local strong solution mentioned in (1.8), and  $\mu, \lambda$  be as in Theorem 1.1. The initial data  $\rho_0 > \varepsilon_0 > 0$ ,  $u_0$  and  $F_0$  as in Theorem 1.1,  $T^*$  is the maximal existence time of the solution. If  $T^* < \infty$ , then we have

$$(1.10) \quad \lim_{T \uparrow T^*} \sup \{ \|\rho(t)\|_{L^\infty(0, T; L^\infty(\Omega))} + \|F\|_{L^\infty(0, T; L^\infty(\Omega))} \} = \infty.$$

**Remark 1.3.** According to (2.7), the condition (1.10) implies that, if  $\|\rho\|_{L^\infty(0, T, L^\infty(\Omega))} = \infty$ , or equivalently  $\det F = 0$ , the mass accumulate at one point or the material volume compress to zero. If  $\|F\|_{L^\infty(0, T, L^\infty(\Omega))} = \infty$ , then the deformation leads to the blow-up phenomenon.

If  $\|\rho(t)\|_{L^\infty(0, T; L^\infty(\Omega))}$  and  $\|F\|_{L^\infty(0, T; L^\infty(\Omega))}$  are bounded, we can obtain a high integrability of velocity, which can be used to control the nonlinear term (See Lemma 3.2). The difficulty is to control the density and deformation, which satisfy transport equations. To do this, it requires the velocity is bounded in  $L^1(0, T; W^{1, \infty}(\Omega))$ . On the other hand, we have to obtain some priori bounds for  $\nabla \rho$  and  $\nabla F$  to prove  $u \in L^1(0, T; W^{1, \infty}(\Omega))$ , furthermore, the elasticity term  $\nabla \cdot (\frac{1}{\det F} F F^T)$  in the momentum equation will bring extra difficulty to us. Thanks to the structure of the equations, we get the cancelation to the derivatives of  $\rho, F$  during our computation, which brings us the desired result. Moreover, we get the result without the restriction on  $\rho^{-1}$ , it is an advantage compared with Sun-Wang-Zhang [20].

In this paper, we follow the line of Sun-Wang-Zhang [19, 20], write

$$(1.11) \quad L \triangleq \mu \Delta + (\lambda + \mu) \nabla \operatorname{div}$$

then, the Lamé operator  $L$  is an elliptical operate.

Generally speaking, the fluid is driven by three kinds of forces which present different mechanisms. They are gradient of pressure, elastic deformation, and inertial force. According to these mechanisms, we can decompose the velocity field to several parts. Here,

since the pressure term and the elasticity term have similar form in the equation, we put them together and decompose velocity field into two parts, that is

$$(1.12) \quad u = v + w,$$

with

$$(1.13) \quad Lv = \nabla p - \nabla \cdot \left( \frac{1}{\det F} F F^T \right),$$

and

$$(1.14) \quad Lw = \rho \dot{u},$$

where

$$\dot{u} = \partial_t u + u \cdot \nabla u.$$

By the decomposition, in order to obtain the regularity of  $u$ , it is sufficient to consider the regularity of  $v$  and  $w$ , which admit the above elliptic equations.

Our paper is organized as following. In Section 2, we shall present some preliminaries. Section 3 is to present some priori estimates. The proof of Theorem 1.2 should be presented in Section 4.

## 2. PRELIMINARIES

Consider the following boundary value problem for the Lamé operator  $L$

$$(2.1) \quad \begin{cases} \mu \Delta U + (\mu + \lambda) \nabla \operatorname{div} U = F, & \text{in } \Omega \\ U(x) = 0, & \text{on } \partial\Omega \end{cases}$$

where  $\Omega$  is  $\mathbb{R}^3$  or a bounded domain in  $\mathbb{R}^3$ . We present the following Lemmas, see [19, 20] for details.

**Lemma 2.1.** *Let  $q \in (1, \infty)$  and  $U$  be a solution of (2.1). There exists a constant  $C$  depending only on  $\lambda, \mu, q$  and  $\Omega$  such that the following estimates hold.*

(1) *If  $F \in L^q(\Omega)$ , then*

$$(2.2) \quad \begin{cases} \|D^2 U\|_{L^q(\mathbb{R}^3)} \leq C \|F\|_{L^q(\mathbb{R}^3)}, \\ \|U\|_{W^{2,q}(\Omega)} \leq C \|F\|_{L^q(\Omega)}; \end{cases} \quad \text{if } \Omega \text{ is a bounded domain.}$$

(2) *If  $F \in W^{-1,q}(\Omega)$  (i.e.,  $F = \operatorname{div} f$  with  $f = (f_{ij})_{3 \times 3}$ ,  $f_{ij} \in L^q(\Omega)$ ), then*

$$(2.3) \quad \begin{cases} \|DU\|_{L^q(\mathbb{R}^3)} \leq C \|f\|_{L^q(\mathbb{R}^3)}, \\ \|U\|_{W^{1,q}(\Omega)} \leq C \|f\|_{L^q(\Omega)}; \end{cases} \quad \text{if } \Omega \text{ is a bounded domain.}$$

(3) *If  $F = \operatorname{div} f$  with  $f_{ij} = \partial_k H_{ij}^k$  and  $h_{ij}^k \in W_0^{1,q}(\Omega)$  for  $i, j, k = 1, 2, 3$ , then*

$$(2.4) \quad \|U\|_{L^q(\Omega)} \leq C \|h\|_{L^q(\Omega)}.$$

**Lemma 2.2.** *If  $F = \operatorname{div} f$  with  $f = (f_{ij})_{3 \times 3}$ ,  $f_{ij} \in L^\infty(\Omega) \cap L^2(\Omega)$ , then  $\nabla U \in BMO(\Omega)$  and there exists a constant  $C$  depending only on  $\lambda, \mu$  and  $\Omega$  such that*

$$(2.5) \quad \|\nabla U\|_{BMO(\Omega)} \leq C(\|f\|_{L^\infty(\Omega)} + \|f\|_{L^2(\Omega)}).$$

**Lemma 2.3.** *Let  $\Omega = \mathbb{R}^3$  or be a bounded Lipschitz domain and  $f \in W^{1,q}(\Omega)$  with  $q \in (3, \infty)$ . There exists a constant  $C$  depending on  $q$  and the Lipschitz property of  $\Omega$  such that*

$$(2.6) \quad \|f\|_{L^\infty(\Omega)} \leq C(1 + \|f\|_{BMO(\Omega)}) \ln(e + \|\nabla f\|_{L^q(\Omega)}).$$

The following lemma is well-known.

**Lemma 2.4.** *Let  $\rho, F$  as defined before, and  $\rho_0$  is the initial data of  $\rho$ , then*

$$(2.7) \quad \rho \cdot \det F = \rho_0.$$

In the following, we use the notation, for the matrix  $(A)_{3 \times 3}$  and  $(B)_{3 \times 3}$

$$A : B = A_{ij} B_{ij},$$

where summation applied to terms with repeated index.

### 3. PRIORI ESTIMATE

By the standard energy estimates, we have

**Lemma 3.1.** *(Priori estimate)*

$$\|\rho(t)\|_{L^1(\Omega)} = \|\rho_0\|_{L^1(\Omega)},$$

$$(3.1) \quad \left\| \sqrt{\frac{\rho}{\rho_0}} F \right\|_{L^2(\Omega)} + \|\rho(t)\|_{L^\gamma(\Omega)}^\gamma + \|\rho|u|^2(t)\|_{L^1(\Omega)} + \|\nabla u\|_{L^2((0,t) \times \Omega)}^2 \\ \leq C(\|\rho_0\|_{L^\gamma(\Omega)}^\gamma + \|\rho_0|u_0|^2\|_{L^1(\Omega)} + \|F_0\|_{L^2(\Omega)}).$$

**Lemma 3.2.** *Assume the initial data as in Theorem 1.2,  $7\mu > \lambda$ , the density  $\rho$  and deformation tensor  $F$  satisfy*

$$(3.2) \quad \|\rho\|_{L^\infty(0,T;L^\infty(\Omega))} + \|F\|_{L^\infty(0,T;L^\infty(\Omega))} < \infty,^1$$

*then, there exists  $r \in (3, 6)$ , such that  $\rho|u|^r \in L^\infty(0, T; L^1(\Omega))$ , with*

$$(3.3) \quad \|\rho|u|^r\|_{L^\infty(0,T;L^1(\Omega))} \leq C.$$

*Here  $C$  depends on  $T$ ,  $\|\rho\|_{L^\infty(\Omega)}$ ,  $\|F\|_{L^\infty(\Omega)}$  and initial data.*

*Proof.* Multiplying the equation (1.7)<sub>2</sub> by  $r|u|^{r-2}u$  and integrating the resulting equation on  $\Omega$  to obtain

$$(3.4) \quad \frac{d}{dt} \int_{\Omega} \rho|u|^r dx + \int_{\Omega} r|u|^{r-2}(\mu|\nabla u|^2 + (\lambda + \mu)(\operatorname{div} u)^2) \\ + r(r-2)(\mu|u|^{r-2}|\nabla|u||^2 + (\lambda + \mu)\operatorname{div} u|u|^{r-3}u \cdot \nabla|u|)dx \\ = \int_{\Omega} rP(\rho)\operatorname{div}(|u|^{r-2}u)dx - \int_{\Omega} r\nabla(|u|^{r-2}u) : \left(\frac{1}{\det F} FF^T\right)dx.$$

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<sup>1</sup>Recall Lemma 2.4 and  $\rho_0 > \varepsilon_0 > 0$ , this condition implies  $0 < c_0 < \det F < \infty$ , a.e.

By using the fact  $|\nabla u| \geq |\nabla|u||$ , the term in second integrand can be estimated by

$$\begin{aligned}
 (3.5) \quad & r|u|^{r-2}[\mu|\nabla u|^2 + (\lambda + \mu)(\operatorname{div} u)^2 + (r-2)\mu|\nabla|u||^2 - (\lambda + \mu)(r-2)|\nabla|u||\operatorname{div} u] \\
 & \geq r|u|^{r-2}[\mu|\nabla u|^2 + (r-2)(\mu - (\lambda + \mu)\frac{r-2}{4})|\nabla|u||^2] \\
 & \geq C|u|^{r-2}|\nabla u|^2.
 \end{aligned}$$

The pressure term

$$(3.6) \quad \int_{\Omega} P(\rho) \operatorname{div}(|u|^{r-2}u) dx \leq C \int_{\Omega} \rho^{\frac{r-2}{2r}} |u|^{r-2} |\nabla u| dx,$$

and the elasticity term

$$(3.7) \quad \int_{\Omega} |\nabla(|u|^{r-2}u) : \frac{1}{\det F} FF^T| dx \leq C \int_{\Omega} (|u|^{r-2} |\nabla u| \rho^{\frac{r-2}{2r}}) dx.$$

By using

$$(3.8) \quad \int_{\Omega} \rho^{\frac{r-2}{2r}} |u|^{r-2} |\nabla u| dx \leq \varepsilon \int_{\Omega} |u|^{r-2} |\nabla u|^2 dx + \frac{C}{\varepsilon} \left( \int_{\Omega} \rho |u|^r dx \right)^{\frac{r-2}{r}},$$

then, (3.4)-(3.7) imply the desired estimate.  $\square$

**Proposition 3.3.** *Under the assumption (3.2), then we have*

$$(3.9) \quad \|\nabla w\|_{L^\infty(0,T,L^2(\Omega))}, \|\rho^{\frac{1}{2}} \partial_t w\|_{L^2(0,T,\Omega)}, \|\nabla^2 w\|_{L^2(0,T,\Omega)} \leq C,$$

where  $C$  is constant depends on  $\|\rho\|_{L^\infty(0,T,L^\infty(\Omega))}$ ,  $\|F\|_{L^\infty(0,T,L^\infty(\Omega))}$  and the initial data.

*Proof.* Using the momentum equation, we get

$$(3.10) \quad \begin{cases} \rho \partial_t w - Lw = \rho G, & \text{in } [0, T) \times \Omega, \\ w(t, x) = 0, & \text{on } [0, T) \times \partial\Omega, \quad w(0, x) = w_0(x) \quad \text{in } \Omega, \end{cases}$$

where

$$\begin{aligned}
 (3.11) \quad G &= -u \operatorname{div} u - L^{-1} \nabla(\partial_t P) + L^{-1} \nabla \partial_t \left( \frac{1}{\det F} FF^T \right) \\
 &= -u \operatorname{div} u + L^{-1} \nabla \operatorname{div}(Pu) + L^{-1} \nabla [(\rho P'(\rho) - P) \operatorname{div} u] \\
 &\quad + L^{-1} \nabla [\nabla u \cdot \left( \frac{1}{\det F} FF^T \right) + \frac{1}{\det F} FF^T (\nabla u)^T - \operatorname{div}(u \otimes \frac{1}{\det F} FF^T)].
 \end{aligned}$$

Multiplying the equation with  $\partial_t w$  and integrating over  $\Omega$ , with the Hölder inequality, we get

$$(3.12) \quad \frac{d}{dt} \int_{\Omega} \mu |\nabla w|^2 + (\lambda + \mu) |\operatorname{div} w|^2 dx + \frac{1}{2} \int_{\Omega} \rho |\partial_t w|^2 dx \leq \frac{1}{2} \|\sqrt{\rho} G\|_{L^2(\Omega)}^2.$$

Now, we shall estimate  $\|\sqrt{\rho} G\|_{L^2(\Omega)}^2$  term by term.

$$\begin{aligned}
 (3.13) \quad & \|\sqrt{\rho} u \operatorname{div} u\|_{L^2(\Omega)} \leq C \|\rho^{\frac{1}{r}}\|_{L^r(\Omega)} \|\nabla u\|_{L^{\frac{2r}{r-2}}(\Omega)} \\
 & \leq C(\varepsilon) \|\nabla w\|_{L^2(\Omega)} + \varepsilon \|\nabla^2 w\|_{L^2(\Omega)} + C,
 \end{aligned}$$

Here  $2 \leq r < 6$ , and we used the interpolation inequality

$$(3.14) \quad \|\cdot\|_{L^r(\Omega)} \leq C(\varepsilon) \|\cdot\|_{L^2(\Omega)} + \varepsilon \|\nabla \cdot\|_{L^2(\Omega)}.$$

From the estimates for Lamé operator and the energy estimates Lemma 3.1, we have

$$(3.15) \quad \|\sqrt{\rho}L^{-1}\nabla\operatorname{div}(Pu)\|_{L^2(\Omega)} \leq C\|Pu\|_{L^2(\Omega)} \leq C\|\sqrt{\rho}u\|_{L^2(\Omega)} \leq C,$$

and

$$(3.16) \quad \begin{aligned} & \|\sqrt{\rho}L^{-1}\nabla[\nabla u \cdot (\frac{1}{\det F}FF^T) + \frac{1}{\det F}FF^T(\nabla u)^T - \nabla \cdot (u \otimes \frac{1}{\det F}FF^T)]\|_{L^2(\Omega)} \\ & \leq C\|\sqrt{\rho}\|_{L^3(\Omega)}\|L^{-1}\nabla[\nabla u \cdot (\frac{1}{\det F}FF^T) + \frac{1}{\det F}FF^T(\nabla u)^T]\|_{L^6(\Omega)} \\ & \quad + C\|L^{-1}\nabla[\nabla \cdot (u \otimes \frac{1}{\det F}FF^T)]\|_{L^2(\Omega)} \\ & \leq C\|\nabla u\|_{L^2(\Omega)} + C. \end{aligned}$$

Similarly, we have

$$(3.17) \quad \|\sqrt{\rho}L^{-1}\nabla[(\rho P' - P)\operatorname{div}u]\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}.$$

Then, the results (3.9) follows from (3.11)-(3.16) and energy estimates Lemma 3.1.  $\square$

**Corollary 3.4.** *Under the assumption of Lemma 3.2, we have*

$$(3.18) \quad \|\nabla u\|_{L^\infty(0,T,L^2)}, \|\nabla u\|_{L^2(0,T,L^q(\Omega))} \leq C,$$

for any  $q \in [2, 6]$ .

#### 4. PROOF OF THEOREM 1.2

**Proposition 4.1.** *Suppose  $T^* < \infty$  is the maximal existence time,  $\forall T, 0 \leq T < T^*$  and  $3 < q < 6$ , if*

$$(4.1) \quad \|\rho, \rho^{-1}\|_{L^\infty(0,T,L^\infty(\Omega))} + \|F\|_{L^\infty(0,T,L^\infty(\Omega))} < \infty,$$

then, we obtain

$$(4.2) \quad \|\nabla^2 w\|_{L^2(0,T;L^q(\Omega))} \leq C,$$

and

$$(4.3) \quad \int_0^T \|\nabla u\|_{L^\infty} ds < \infty.$$

*Proof.* We first estimate  $\|\nabla^2 w\|_{L^2(0,T;L^q(\Omega))}$ . By Lemma 2.1 and (1.14), we have

$$(4.4) \quad \|\nabla^2 w\|_{L^q(\Omega)} \leq \|\rho \dot{u}\|_{L^q(\Omega)}.$$

Noting,

$$(4.5) \quad \rho \dot{u} - Lu + \nabla P = \nabla \cdot (\frac{1}{\det F}FF^T),$$

taking  $\partial_t$  to this equation, we get

$$(4.6) \quad \rho_t \dot{u} + \rho \dot{u}_t - Lu_t + \nabla P_t = \nabla \cdot (\frac{1}{\det F}FF^T)_t.$$

Applying  $u \otimes$  and divergence to (4.5), we have

$$(4.7) \quad \nabla \cdot (\rho u) \dot{u} + \rho u \cdot \nabla \dot{u} - \nabla \cdot (u \otimes Lu) + \nabla \cdot (u \otimes \nabla P) = \nabla \cdot [u \otimes \nabla \cdot (\frac{1}{\det F}FF^T)].$$

Adding (4.6) and (4.7) together, we get

$$\begin{aligned}
 (4.8) \quad & \rho \dot{u}_t + \rho u \cdot \nabla \dot{u} + \nabla P_t + \nabla \cdot [u \otimes \nabla P] \\
 &= \mu [\Delta u_t + \nabla \cdot (u \otimes \Delta u)] + (\lambda + \mu) [\nabla \operatorname{div} u_t + \operatorname{div}(u \otimes \nabla \operatorname{div} u)] \\
 &\quad + \nabla \cdot \left[ \left( \frac{1}{\det F} F F^T \right)_t + u \otimes \nabla \cdot \left( \frac{1}{\det F} F F^T \right) \right].
 \end{aligned}$$

Multiplying above equation by  $\dot{u}$  and integrating on  $\Omega$ , we get

$$\begin{aligned}
 (4.9) \quad & \frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho |\dot{u}|^2 dx - \mu \int_{\Omega} (\Delta u_t + \operatorname{div}(u \otimes \Delta u)) \cdot \dot{u} dx \\
 &\quad - (\lambda + \mu) \int_{\Omega} (\nabla \operatorname{div} u_t + \operatorname{div}(u \otimes \nabla \operatorname{div} u)) \cdot \dot{u} dx \\
 &= \int_{\Omega} P_t \operatorname{div} \dot{u} dx + \int_{\Omega} u \cdot \nabla \dot{u} \cdot \nabla p dx \\
 &\quad + \int_{\Omega} \nabla \cdot \left[ \left( \frac{1}{\det F} F F^T \right)_t + u \otimes \nabla \cdot \left( \frac{1}{\det F} F F^T \right) \right] \cdot \dot{u} dx.
 \end{aligned}$$

The estimates of the second and the third terms on the left hand side, as well as the first and the second terms on the right hand side in (4.9), are the same as in [19, 20]. For completeness, we give a brief proof as following

$$\begin{aligned}
 (4.10) \quad & - \int_{\Omega} (\Delta u_t + \operatorname{div}(u \otimes \Delta u)) \cdot \dot{u} dx = \int_{\Omega} \nabla u_t : \nabla \dot{u} + u \otimes \Delta u : \nabla \dot{u} dx \\
 &= \int_{\Omega} |\nabla \dot{u}|^2 - \nabla(u \cdot \nabla u) : \nabla \dot{u} + u \times \Delta u : \nabla \dot{u} dx \\
 &= \int_{\Omega} [|\nabla \dot{u}|^2 - (\nabla u \nabla u) : \nabla \dot{u} + ((u \cdot \nabla) \nabla \dot{u}) : \nabla u \\
 &\quad - (\nabla u \nabla \dot{u}) : \nabla u - ((u \cdot \nabla) \nabla \dot{u}) : \nabla u] dx \\
 &\geq \int_{\Omega} \left[ \frac{3}{4} |\nabla \dot{u}|^2 - C |\nabla u|^4 \right] dx,
 \end{aligned}$$

and

$$\begin{aligned}
 (4.11) \quad & - \int_{\Omega} (\nabla \operatorname{div} u_t + \operatorname{div}(u \otimes \nabla \operatorname{div} u)) \cdot \dot{u} dx \\
 &= \int_{\Omega} [|\operatorname{div} \dot{u}|^2 - \operatorname{div} \dot{u} \nabla u : (\nabla u)^T - \operatorname{div} u (\nabla \dot{u})^T : \nabla u + \operatorname{div} \dot{u} (\operatorname{div} u)^2] dx \\
 &\geq \int_{\Omega} \left[ \frac{1}{2} |\operatorname{div} \dot{u}|^2 - \frac{1}{4} |\nabla \dot{u}|^2 - C |\nabla u|^4 \right] dx.
 \end{aligned}$$

We continue to estimate the pressure term.

$$\begin{aligned}
 (4.12) \quad & \int_{\Omega} P_t \operatorname{div} \dot{u} + (u \cdot \nabla \dot{u}) \cdot \nabla P dx \\
 &= \int_{\Omega} -\rho P'(\rho) \operatorname{div} u \operatorname{div} \dot{u} + P [\operatorname{div} u \operatorname{div} \dot{u} - (\nabla u)' : (\nabla \dot{u})] dx \\
 &\leq C \|\nabla u\|_{L^2(\Omega)} \|\nabla \dot{u}\|_{L^2(\Omega)} \leq C \|\nabla \dot{u}\|_{L^2(\Omega)}.
 \end{aligned}$$



Next, we estimate the elasticity term  $\int_{\Omega} \nabla \cdot [(\frac{1}{\det F} FF^T)_t + u \otimes \nabla \cdot (\frac{1}{\det F} FF^T)] \cdot \dot{u} dx$ . From (1.7), we get

$$\begin{aligned}
 (4.13) \quad & \int_{\Omega} \nabla \cdot [(\frac{1}{\det F} FF^T)_t + u \otimes \nabla \cdot (\frac{1}{\det F} FF^T)] \cdot \dot{u} dx \\
 &= \int_{\Omega} \nabla \cdot [\nabla u \cdot (\frac{1}{\det F} FF^T) + \frac{1}{\det F} FF^T (\nabla u)^T - \frac{1}{\det F} FF^T (\nabla \cdot u)] \dot{u} dx \\
 &+ \int_{\Omega} \nabla \cdot [u \otimes \nabla \cdot (\frac{1}{\det F} FF^T) - u \cdot \nabla (\frac{1}{\det F} FF^T)] \dot{u} dx
 \end{aligned}$$

Since  $\|\rho\|_{L^\infty(0,T;L^\infty(\Omega))}, \|F\|_{L^\infty(0,T;L^\infty(\Omega))} < \infty$ , we get the bound for the first three terms on the right hand side of the above equation as following

$$(4.14) \quad \left| \int_{\Omega} \nabla \cdot [\nabla u \cdot (\frac{1}{\det F} FF^T) + \frac{1}{\det F} FF^T (\nabla u)^T - \frac{1}{\det F} FF^T (\nabla \cdot u)] \dot{u} dx \right| \leq C \int_{\Omega} |\nabla u| |\nabla \dot{u}| dx.$$

As to the last term on the right hand side of (4.13), we have a cancelation

$$\begin{aligned}
 (4.15) \quad & \nabla \cdot [u \otimes \nabla \cdot (\frac{1}{\det F} FF^T) - u \cdot \nabla (\frac{1}{\det F} FF^T)] \cdot \dot{u} \\
 &= -\partial_i [\partial_j u_i (\frac{1}{\det F} F_{jk} F_{lk})] \dot{u}_l + \partial_i [\partial_j u_j (\frac{1}{\det F} F_{jk} F_{lk})] \dot{u}_l.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 (4.16) \quad & \int_{\Omega} \nabla \cdot [u \otimes \nabla \cdot (\frac{1}{\det F} FF^T) - u \cdot \nabla (\frac{1}{\det F} FF^T)] \cdot \dot{u} dx \\
 &\leq C \|\nabla \dot{u}\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)}.
 \end{aligned}$$

Combining the above estimates, recalling Lemma 3.1, we get

$$(4.17) \quad \frac{d}{dt} \int_{\Omega} \rho |\dot{u}|^2 dx + \int_{\Omega} |\nabla \dot{u}|^2 dx \leq C(1 + \|\nabla u\|_{L^4(\Omega)}^4).$$

Noting that

$$\begin{aligned}
 (4.18) \quad & \|\nabla u\|_{L^4(\Omega)}^4 \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla u\|_{L^6(\Omega)}^3 \\
 &\leq C \|\nabla u\|_{L^6(\Omega)}^2 (\|\nabla u\|_{L^6(\Omega)} + \|\nabla v\|_{L^6(\Omega)}) \\
 &\leq C \|\nabla u\|_{L^6(\Omega)}^2 (1 + \|\dot{u}\|_{L^2(\Omega)}).
 \end{aligned}$$

Substituting this estimate into (4.17) and using Corollary 3.4, we have  $\|\nabla u(t)\|_{L^6(\Omega)}^2 \in L^1(0, T)$ , then conclude by Gronwall's inequality that

$$(4.19) \quad \|\sqrt{\rho} \dot{u}\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla \dot{u}\|_{L^2((0,T) \times \Omega)} \leq C.$$

Recalling (4.4) and Lemma 2.1, (4.2) is verified.

In the following, we shall prove (4.3). Noting that

$$\begin{aligned}
 (4.20) \quad & \|\nabla u\|_{L^1(0,T;L^\infty(\Omega))} \leq C \|u\|_{L^1(0,T;W^{2,q}(\Omega))} \\
 &\leq C(\|v\|_{L^1(0,T;W^{2,q}(\Omega))} + \|w\|_{L^1(0,T;W^{2,q}(\Omega))}),
 \end{aligned}$$

where  $3 < q < 6$ . By Lemma 2.1,  $\|v\|_{W^{2,q}(\Omega)} \leq \|\nabla P\|_{L^q(\Omega)} + \|\nabla \cdot (\frac{1}{\det F} F F^T)\|_{L^q(\Omega)}$ , from the assumption (4.1) and lemma 2.4, we have

$$(4.21) \quad \|\nabla P\|_{L^q(\Omega)} \leq C \|\nabla \rho\|_{L^q(\Omega)},$$

and

$$(4.22) \quad \begin{aligned} & \|\nabla \cdot (\frac{1}{\det F} F F^T)\|_{L^q(\Omega)} \\ & \leq C \|\nabla \rho\|_{L^q(\Omega)} \|F\|_{L^\infty}^2 + C \|\rho\|_{L^\infty(\Omega)} \|\nabla F\|_{L^q(\Omega)} \|F\|_{L^\infty} \\ & \leq C \|\nabla \rho\|_{L^q(\Omega)} + C \|\nabla F\|_{L^q(\Omega)}. \end{aligned}$$

Therefore, recalling (4.2), to bound  $\|\nabla u\|_{L^1(0,T,L^\infty(\Omega))}$ , it suffices to bound  $\|\nabla \rho\|_{L^q(\Omega)}$  and  $\|\nabla F\|_{L^q(\Omega)}$ . Taking gradient to both sides of equation (1.7)<sub>1</sub> and (1.7)<sub>3</sub>, we have

$$(4.23) \quad (\nabla \rho)_t + u \cdot \nabla(\nabla \rho) = -\nabla(\rho \nabla \cdot u) - \nabla u \cdot \nabla \rho,$$

and

$$(4.24) \quad \rho(\nabla F)_t + \rho u \cdot \nabla(\nabla F) = \rho \nabla(\nabla u F) - \rho \nabla u \nabla F.$$

Multiply by  $q\rho|\nabla \rho|^{q-2}\nabla \rho$  and  $q\rho|\nabla F|^{q-2}\nabla F$  to (4.23) and (4.24) respectively, we have

$$(4.25) \quad \begin{aligned} & \frac{d}{dt} \int_{\Omega} \rho |\nabla \rho|^q dx \\ & = -q \int_{\Omega} \nabla(\rho \nabla \cdot u) \rho |\nabla \rho|^{q-2} \nabla \rho dx - q \int_{\Omega} \nabla u \cdot \nabla \rho \rho |\nabla \rho|^{q-2} \nabla \rho dx \\ & \leq C \int_{\Omega} \rho |\nabla u| |\nabla \rho|^q dx + C \int_{\Omega} \rho |\nabla^2 u| |\nabla \rho|^{q-1} dx \\ & \leq C \|\nabla u\|_{L^\infty(\Omega)} \|\rho^{1/q} \nabla \rho\|_{L^q(\Omega)}^q + C \|\nabla^2 u\|_{L^q(\Omega)} \|\rho^{1/q} \nabla \rho\|_{L^q(\Omega)}^{q-1}, \end{aligned}$$

as well as

$$(4.26) \quad \begin{aligned} & \frac{d}{dt} \int_{\Omega} \rho |\nabla F|^q dx \\ & \leq C \int_{\Omega} |\rho \nabla(\nabla u F)| |\nabla F|^{q-2} \nabla F dx + C \int_{\Omega} |\rho \nabla u \nabla F| |\nabla F|^{q-2} \nabla F dx \\ & \leq C \|\nabla^2 u\|_{L^q(\Omega)} \|\rho^{1/q} \nabla F\|_{L^q(\Omega)}^{q-1} + C \|\nabla u\|_{L^\infty(\Omega)} \|\rho^{1/q} \nabla F\|_{L^q(\Omega)}^q \end{aligned}$$

From (4.25) and (4.26), we get

$$(4.27) \quad \begin{aligned} & \frac{d}{dt} (\|\rho^{1/q} \nabla \rho\|_{L^q(\Omega)}^q + \|\rho^{1/q} \nabla F\|_{L^q(\Omega)}^q) \\ & \leq C \|\nabla^2 u\|_{L^q(\Omega)} (\|\rho^{1/q} \nabla \rho\|_{L^q(\Omega)}^{q-1} + \|\rho^{1/q} \nabla F\|_{L^q(\Omega)}^{q-1}) \\ & \quad + C \|\nabla u\|_{L^\infty(\Omega)} (\|\rho^{1/q} \nabla \rho\|_{L^q(\Omega)}^q + \|\rho^{1/q} \nabla F\|_{L^q(\Omega)}^q). \end{aligned}$$

Noting that

$$(4.28) \quad \|\nabla u\|_{L^\infty(\Omega)} \leq \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla w\|_{L^\infty(\Omega)},$$

$$(4.29) \quad \|\nabla^2 u\|_{L^q(\Omega)} \leq \|\nabla^2 v\|_{L^q(\Omega)} + \|\nabla^2 w\|_{L^q(\Omega)},$$

and by Lemma 2.2,

$$(4.30) \quad \begin{aligned} \|\nabla v\|_{L^\infty(\Omega)} &\leq 1 + \|\nabla v\|_{BMO(\Omega)} \ln(e + \|\nabla^2 v\|_{L^q(\Omega)}) \\ &\leq 1 + C \ln(e + \|\nabla^2 v\|_{L^q(\Omega)}). \end{aligned}$$

Then from (4.2), we obtain

$$(4.31) \quad \begin{aligned} \frac{d}{dt}(e + \|\rho^{1/q} \nabla \rho\|_{L^q(\Omega)} + \|\rho^{1/q} \nabla F\|_{L^q(\Omega)}) \\ \leq C[1 + \ln(e + \|\rho^{1/q} \nabla \rho\|_{L^q(\Omega)} + \|\rho^{1/q} \nabla F\|_{L^q(\Omega)})] \\ (e + \|\rho^{1/q} \nabla \rho\|_{L^q(\Omega)} + \|\rho^{1/q} \nabla F\|_{L^q(\Omega)}). \end{aligned}$$

Using Gronwall inequality, we get  $\|\rho^{1/q} \nabla \rho\|_{L^q(\Omega)}$  and  $\|\rho^{1/q} \nabla F\|_{L^q(\Omega)}$  are finite, hence  $\|\nabla^2 v\|_{L^q(\Omega)}$  is bounded.  $\square$

**Remark 4.2.** By (2.7), we have  $\rho^{-1} = \det F / \rho_0$ . When  $\rho_0$  has lower bound  $\varepsilon_0$  and  $F \in L^\infty(\Omega)$ , we have  $\|\rho^{-1}\|_{L^\infty(\Omega)} \leq \frac{\|F\|_{L^\infty(\Omega)}^3}{\varepsilon_0} < \infty$ . Namely, if we restrict  $\|F\|_{L^\infty(0,T,L^\infty(\Omega))} < \infty$ , then  $\|\rho^{-1}\|_{L^\infty(0,T,L^\infty(\Omega))} < \infty$  will hold naturally.

Now we are to prove our main theorem by the contradiction argument.

**Proof of Theorem 1.2.** Suppose the maximal existence time of solution  $T^* < \infty$ , and  $\|\rho\|_{L^\infty(0,T^*,L^\infty(\Omega))} + \|F\|_{L^\infty(0,T^*,L^\infty(\Omega))} < \infty$ . By Remark 4.2 and Proposition 4.1, we obtain

$$\|\nabla u\|_{L^1(0,T^*,L^\infty(\Omega))} < \infty.$$

This is a contradiction with Theorem 1.1.

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